Noise Enhanced Propagation

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We use noise to extend signal propagation in one- and two-dimensional arrays of two-way coupled bistable oscillators. In a numerical model, we sinusoidally force one end of a chain of noisy oscillators. We record a signal-to-noise ratio at each oscillator. We demonstrate that moderate noise significantly extends the propagation of the sinusoidal input. Oscillators far from the input, where noise extends the signal, exhibit a classical stochastic resonance. We obtain similar results with two-dimensional arrays. The simplicity of the model suggests the generality of the phenomenon. [S0031-9007(98)07826-0]

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In the phenomenon of *stochastic resonance* (SR), a nonzero value of noise optimizes the response of a nonlinear system to a deterministic signal [1]. Coupling such a stochastic resonator into an array of similar resonators enhances the effect, producing a spatiotemporal or *array enhanced stochastic resonance* (AESR) [2]. Recent studies have demonstrated that noise can support or sustain propagation in a variety of numerical and experimental nonlinear systems. Jung *et al.* [3] demonstrated that noise can aid the spreading of wave fronts in a numerical model of an excitable medium. Kádár *et al.* [4] established that noise can support traveling waves in a chemical subexcitable medium, a photosensitive Belousov-Zhabotinsky reaction. Löcher *et al.* [5] showed that noise can sustain signal propagation in a chain of coupled diode resonators operated in biased bistable states. Zhang *et al.* [6] demonstrated that, with sufficient coupling, noise can induce undamped signal transmission in a numerical model of a chain of *one-way* coupled bistable elements. Furthermore, Sendiña-Nadal *et al.* [7] introduced random spatial fluctuations in an excitable medium and studied their effects on the propagation of autowaves, while Castelpoggi and Wio [8] recently addressed the problem of local vs global coupling in reaction-diffusion characterizations of "stochastic resonant media."

In this Letter, we establish the robustness of *noise enhanced propagation* (NEP) by realizing it in a simple and generic system, a numerical model of a chain of bistable elements with *two-way* nearest-neighbor coupling. Like AESR, we view NEP as an important, generic, and nontrivial extension of SR, a cooperative phenomenon involving signal, noise, nonlinearity, and coupling. We suggest scaling laws for optimizing NEP as a function of coupling and noise. We also observe NEP in two-dimensional arrays of bistable elements. NEP may be important in biophysical and biochemical processes, especially neural networks, and may be exploited by communication and detection technologies.

In order to demonstrate NEP as broadly as possible, we focus on a simple model. We study a coupled chain of noisy overdamped bistable oscillators. For $n > 1$, the amplitude x_n of the nth oscillator obeys

$$\dot{x}_n = -V'[x_n] + \varepsilon(x_{n-1} - x_n) + \varepsilon(x_{n+1} - x_n) + N_n[t], \tag{1}$$

where the prime denotes the spatial gradient, the dot denotes time differentiation, the coupling $\varepsilon \geq 0$, and the potential energy function characterizing each element is

$$V[x] = -k_1\frac{x^2}{2} + k_2\frac{x^4}{4}, \tag{2}$$

where $k_1, k_2 > 0$ to ensure its bistability, and the height and width of the energy barrier are $h = {k_1}^2/4k_2$ and $w = 2\sqrt{k_1/k_2}$, respectively. With the end of the chain free, we force the *first* element sinusoidally,

$$\dot{x}_1 = -V'[x_1] + \varepsilon(x_2 - x_1) + N_1[t] + A\sin\omega t, \tag{3}$$

and study the propagation of this signal along the chain.

$N_n[t]$ is Gaussian white noise, bandlimited in practice by our integration time step dt (which establishes a nonzero correlation time) to a Nyquist frequency $f_N = 1/(2dt)$. We quantify the noise by its mean squared amplitude or noise power $\sigma^2 = 2Df_N$, where $2D$ is the height of the one-sided noise spectrum. We study the case of incoherent or local noise (uncorrelated from oscillator to oscillator) rather than coherent or global noise (identical at each oscillator).

We numerically integrate the system of stochastic differential Eqs. (1)–(3) using the Euler-Maruyama algorithm [9], with a small time step $dt = T/2^{12}$, where the forcing period $T = 2\pi/\omega$. We then numerically estimate the spectrum, or power spectral density (PSD), of a long time series of each oscillator. After averaging many such PSD's, we estimate the signal-to-noise ratio (SNR) at each oscillator as the ratio of the signal power to the noise power, at the frequency of the forcing, conventionally expressed in decibels (dB). We estimate the noise power by performing a nonlinear fit to the PSD around—but not

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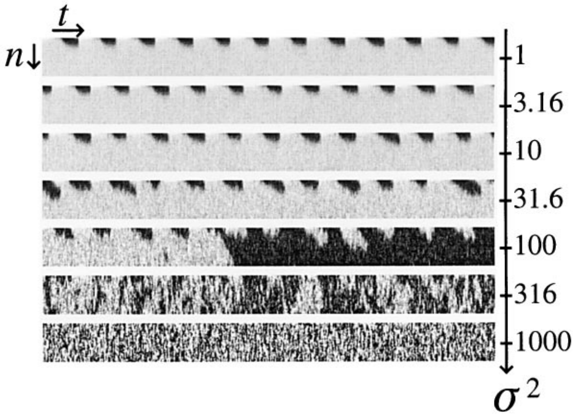

FIG. 1. Spatiotemporal behavior of a chain of 32 overdampled bistable oscillators, sinusoidally forced at one end and subjected to increasing incoherent noise. Parameters are $h = 0.75$, $w = 2.25$, and $A = 5.0$, $\omega = 0.2$, and $\varepsilon = 10$.

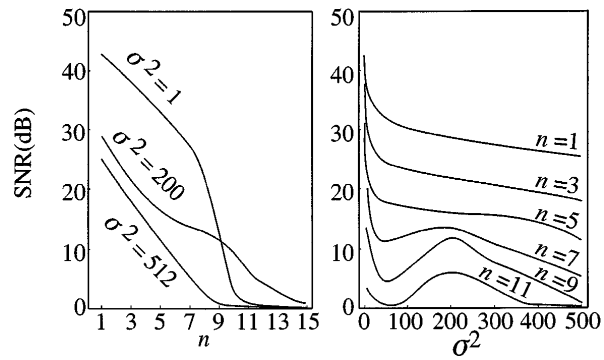

FIG. 2. Plots of SNR versus oscillator number n and SNR versus noise variance σ^2. Intermediate noise results in maximum propagation. Oscillators in the chain where noise extends the signal exhibit stochastic resonance. Smooth curves have been fit to the data to aid the eye. Parameters are the same as Fig. 1 except $\varepsilon = 4$.

including—the forcing frequency. We estimate the signal power by subtracting this noise background from the total power at the forcing frequency. Schematically,

$$\mathrm{SNR} = 10\log_{10}\left[\frac{\text{signal power}}{\text{noise power}}\right]. \tag{4}$$

However, our results are robust with respect to variations in this definition of SNR.

Figure 1 illustrates the system's spatiotemporal behavior in the presence of increasing noise. Each strip represents the evolution of a chain of 32 oscillators, arrayed vertically and evolving horizontally, for 12 forcing periods. The gray scale codes positions $x_n[t]$, with black denoting the left well and white denoting the right well. The first forced oscillator is at the top edge of each strip. With little noise, the sinusoidal signal propagates only a short distance down the chain. Moderate noise extends the propagation, while excessive noise destroys it. Note how an intermediate noise variance of $\sigma^2 = 100$ enables the signal to initiate systemwide events, occasionally causing the entire chain to hop from one well to the other. In fact, we chose our initial operating parameters so that, in the absence of forcing, the spatiotemporal features are large compared to the chain length and forcing period, and large segments of the chain spontaneously flop across the bistable potential barrier. Such a partially correlated medium allows noise to support, sustain, and enhance signal propagation. We parametrize the forcing with amplitude $A = 5.0$ and angular frequency $\omega = 0.2$, and the bistable potential with a barrier height of $h = 0.75$ and barrier width of $w = 2.25$ (corresponding to $k_1 \approx 2.37$ and $k_2 \approx 1.87$). However, NEP is robust with respect to variations in these parameters.

Figure 2 illustrates SNR versus oscillator number for increasing noise variance and SNR versus noise variance for increasing oscillator number. For oscillators near the forcing end, the SNR decreases as the noise increases. But for oscillators farther away, the SNR goes through

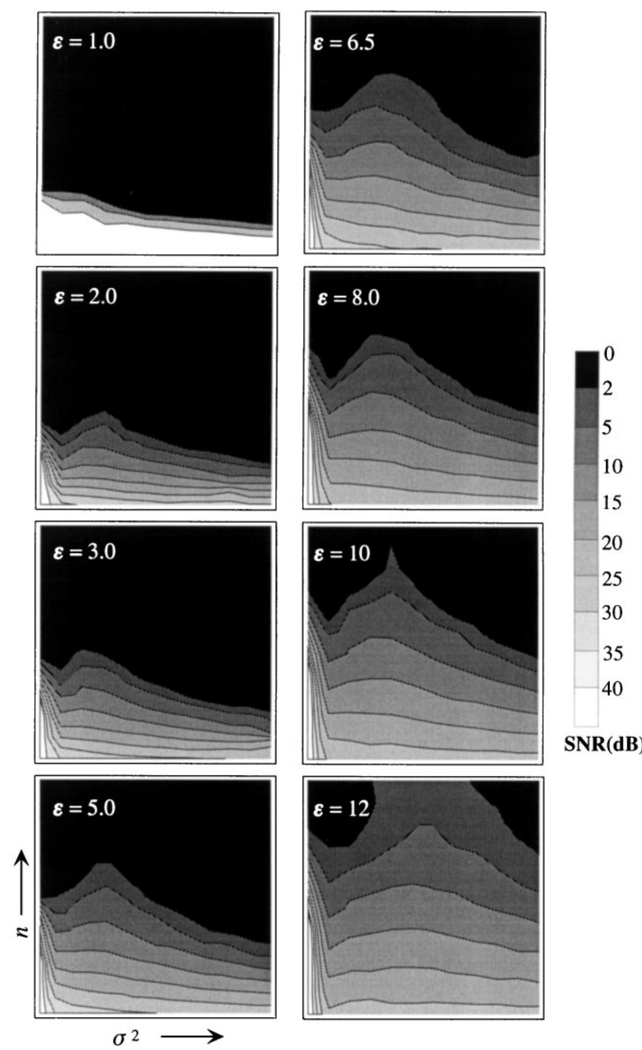

FIG. 3. Smoothed contour plots of SNR versus oscillator number n versus noise variance σ^2, for increasing coupling. SNR contours are accurate to about ± 1 dB. Other parameters are the same as Fig. 1. (Here we employ a larger integration time step of $dt = T/2^{10}$.)

a local maximum as the noise increases—the signature of a classic stochastic resonance. Oscillators near the forcing end do not need help from the noise, as the forcing amplitude there is large ($A > h$); however, oscillators far from the forcing do need help from the noise, as the signal there is attenuated. For a given noise power, SNR decreases with oscillator number, downstream along the chain. Defining the *propagation length* as the number of oscillators (or distance along the chain) for which the SNR exceeds a certain cutoff, say, 1 dB (which is roughly the uncertainty in our numerics), we observe that the propagation length is longest for moderate noise.

This data can be succinctly combined into a contour plot of SNR versus noise variance versus oscillator number. Figure 3 presents a series of such contour plots, for increasing coupling. The gray scale codes SNR, with white indicating large SNR (>40 dB) and black indicating small SNR (<1 dB). The bottom-left corners represent the peak SNR of the noiseless first oscillator. The SNR decreases everywhere away from these corners. However, the distinct bulges in the contours, where regions of large SNR extend toward the free end of the chain, are the signatures of NEP. (A 1 dB contour traces out the propagation length as a function of noise.) At sufficiently large coupling, the SNR extensions reach the end of the chain, indicating that noise and coupling have succeeded in sustaining the signal throughout its length. Note how both the extent and the position of these SNR extensions increase with increasing coupling.

Figure 4 summarizes the scaling of optimal noise and maximum propagation length with coupling. Both appear to scale as the square root of the coupling. This reflects the fact that the correlation length of spatiotemporal features for a local and linearly coupled array scales this way [10]. As the coupling increasingly binds adjacent oscillators together in the same well, the noise variance must increase correspondingly in order to force such correlated oscillators across the bistable potential barrier.

It is instructive to compare these results for two-way (bidirectional) bistable chains with the behavior of one-way (unidirectional) bistable chains recently reported by Zhang *et al.* [6]. For a one-way bistable chain driven at one end, if forcing and noise cooperate to flip the first site (from one well to the other), sufficient coupling propagates the flip to the other end with probability one. When the forcing is just below the threshold for the first site, the requisite noise is small enough not to interfere with the propagation of the flip. Consequently, a one-way bistable chain readily

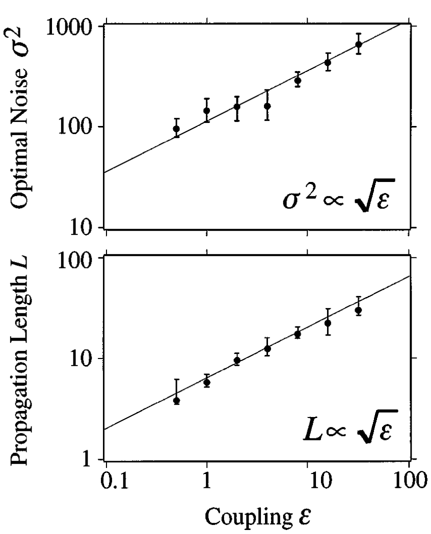

FIG. 4. Scaling of optimal noise σ^2 and maximum propagation length L with coupling ε. Uncertainty bars reflect the averaging of many PSD's of long noisy time series.

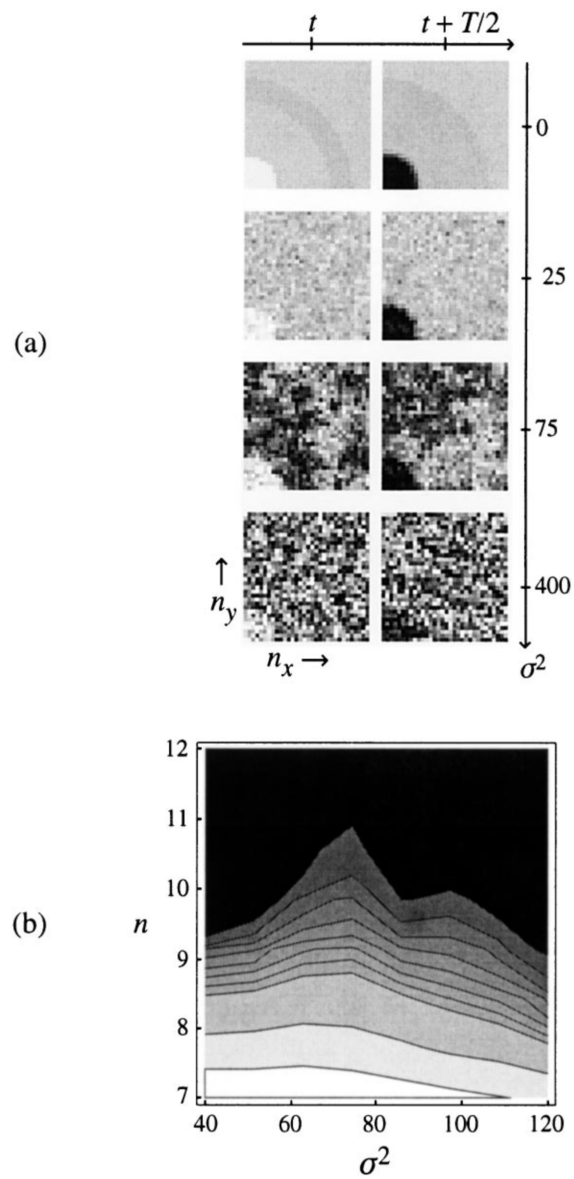

FIG. 5. Examples of (a) spatiotemporal behavior and (b) SNR response at $(n_x, n_y) = (n, n)$ for a two-dimensional array of 32×32 bistable oscillators, with free boundaries, forced from a quarter circle of sites at the bottom-left (0, 0) corner of the array. Parameters are the same as Fig. 1 except $\varepsilon = 1$.

propagates flips, like a line of tumbling dominoes, and for sufficiently large coupling, SR-initiated avalances facilitate undamped signal transmission. A similar avalanchelike propagation mechanism also characterizes the system investigated by Löcher *et al.* [5]. In that experiment, a symmetry-breaking global bias forces kinks to travel towards one boundary, after which all of the resonators must be reset by an opposite global bias, thereby preventing continuous signal transmission. By contrast, the two-way *unbiased* bistable array studied here is not susceptible to avalanche propagation and is well suited to *continuous* information transmission. In our system, NEP results when stochastic resonances at intermediate sites (which are near threshold) extend the influence of the input forcing. Noise, nonlinearity, and forcing (mediated by coupling) cooperate to make this possible.

NEP is not confined to *chains* of bistable elements. We have also established the effect in two-dimensional arrays. We studied a square array of overdampled oscillators, each of which was coupled to its four nearest neighbors. With free boundaries, we sinusoidally forced a circular region of oscillators and monitored the propagation of the resulting circular wave fronts. As in the one-dimensional case, moderate noise extended the propagation length. Figure 5 summarizes the spatiotemporal behavior and SNR response for the two-dimensional array. Note how an intermediate noise of 75 dB corresponds to *both* maximum propagation length *and* significant spatiotemporal organization.

Thus, we have demonstrated that *noise can significantly extend the propagation of signals in arrays of two-way coupled bistable elements.* Like AESR, NEP is an important generalization of SR. Indeed, we believe that the phenomenon of noise-enhanced propagation transcends our intentionally simple model and may be important in both nature and technology.

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